Wave packet through a dispersive medium

Dispersion means wave velocity depends on frequency. The group velocity is the velocity with which a pattern progresses. This is the velocity at which information is carried. Note I use $k = 2\pi/\lambda$, $\omega = 2\pi f$.

Phase velocity

$$\mathrm{v} = \frac{\omega}{k}$$

Group velocity

$$\mathrm{v}_g = \frac{d\omega}{dk} = \frac{d(k\mathrm{v})}{\mathrm{dk}} = \mathrm{v} + k\frac{d\mathrm{v}}{d\lambda}\frac{d\lambda}{dk} = \mathrm{v} - \lambda\frac{d\mathrm{v}}{d\lambda}$$

Now consider a Gaussian function of width σ, modulating a cosine wave of spatial frequency k_0. The displacement is

$$y = G(x)\cos(k_0 x); \quad G = e^{-x^2/\sigma^2}$$

Fourier transform

$$\int_{-\infty}^{\infty} G(x)\cos(k_0 x)e^{-ikx}dx = \int_{-\infty}^{\infty} G(x)\frac{1}{2}\left\{e^{ik_0 x} + e^{-ik_0 x}\right\}e^{-ikx}dx$$
$$= \frac{1}{2}g(k + k_0) + \frac{1}{2}g(k - k_0)$$

where $g(k) = \int G(x)e^{-ikx}dx$ is the transform of G(x). When the function g is even then we can just use the positive frequencies, and omit the factor 2. So for our case we have the F.T. of our wave packet is just the Fourier transform of the Gaussian envelope, but replacing k with $k - k_0$. Now the FT of a Gaussian is

$$g(k) = \int_{-\infty}^{\infty} e^{-x^2/\sigma^2}e^{-ikx}dx = \int_{-\infty}^{\infty} e^{-x^2/\sigma^2 - ikx + k^2\sigma^2/4}e^{-k^2\sigma^2/4}dx$$
$$= e^{-k^2\sigma^2/4}\int_{-\infty}^{\infty} e^{-(x/\sigma + ik\sigma/2)^2}dx$$

Using

$$\int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}; \quad z = \left(\frac{x}{\sigma} + ik\sigma/2\right)$$

we get

$$g(k) = \sigma\sqrt{\pi}e^{-k^2\sigma^2/4}$$

By inverse transform we have a displacement at t = 0

$$y = \int_{-\infty}^{\infty} g(k-k_0)e^{ikx}dk = e^{ik_0x}\int_{-\infty}^{\infty} g(k-k_0)e^{i(k-k_0)x}dk = G(x)e^{ik_0x}$$

As the wave propagates each spatial frequency is associated with a temporal frequency and velocity, so that

$$y = \int_{-\infty}^{\infty} g(k-k_0)e^{i(kx-\omega t)}dk$$

To allow for dispersion write

$$\begin{aligned}\omega &= \omega_0 + (k-k_0)\frac{\partial\omega}{\partial k} + \frac{1}{2}(k-k_0)^2\frac{\partial^2\omega}{\partial k^2}\\ &= \omega_0 + (k-k_0)\alpha + \frac{1}{2}(k-k_0)^2\beta\end{aligned}$$

$$y = \int_{-\infty}^{\infty} g(k-k_0)e^{i\left(kx-\omega_0t-\alpha(k-k_0)t-\frac{1}{2}\beta(k-k_0)^2\right)}dk$$

i.e. $$= \int_{-\infty}^{\infty}\sigma\sqrt{\pi}e^{-(k-k_0)^2\sigma^2/4}e^{i\left(kx-\omega_0t-\alpha(k-k_0)t-\frac{1}{2}\beta(k-k_0)^2\right)}dk$$

$$= e^{i(k_0x-\omega_0t)}\int_{-\infty}^{\infty}\sigma\sqrt{\pi}e^{-(k-k_0)^2/4\left(\sigma^2-i2\beta t\right)}e^{i\left((k-k_0)(x-\alpha t)\right)}d(k-k_0)$$

The integration is achieved by comparison with the inverse FT of Gaussians, and we get

$$y = e^{i(k_0x-\omega_0t)}e^{-\frac{(x-\alpha t)^2}{\left(\sigma^2-i2\beta t\right)}} = e^{i(k_0x-\omega_0t)}e^{-\frac{(x-\alpha t)^2}{\left(\sigma^4+4\beta^2t^2\right)}\left(\sigma^2-i2\beta t\right)}$$

i.e. a Gaussian whose width is a function of time

$$width = \left(\frac{\left(\sigma^4+4\beta^2t^2\right)}{\sigma^2}\right)^{\frac{1}{2}}$$